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Chaos and stochastic modelling of dissipation in a quantum kicked top

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Abstract. The behaviour of a quantum kicked top subject to a dissipation mechanism is considered. Following Gisin and Percival (1992 *J. Phys. A: Math. Gen.* 25 5677) the dissipation is described by means of a diffusion process acting on the vector state; thus the dynamics is given by the stochastic Schrödinger equation. This approach allows one to treat numerically much larger quantum systems than with the use of a density operator. The calculations presented demonstrate an essential similarity between the classical and quantum counterparts of the top, the better the similarity the greater the quantum size of the problem. This concerns both the regular and chaotic behaviour of the top.

1. Introduction

In the classical theory of chaotic dynamical systems both Hamiltonian and dissipative systems are extensively investigated. The situation is completely different in the quantum theory where almost exclusively Hamiltonian cases are treated. This results from the fact that dissipative quantum dynamics is usually described as a reduced dynamics of the system coupled to its environment. Due to a non-Hamiltonian dynamics a state of the system must be given by a density operator. This implies an important consequence for numerical computations: the size of the evolution matrix for an N-level dissipative system is $N^2 \times N^2$ whereas for the corresponding Hamiltonian system it is an $N \times N$ matrix.

A typical dissipative system treated in the literature [1] is any well known Hamiltonian quantum system with some mechanism modelling the dissipation. Examples include a kicked oscillator with damping [2], a damped kicked rotor [3, 4] or a damped kicked top [5, 6]. For such simple models it was possible to obtain many interesting results, either exact or numerical. For example, it was shown that in the semi-classical limit ($\hbar \to 0$) the quantum map reduces to the classical one with first-order corrections having the form of noise terms [2, 3].

In the present paper we adopt as a model the well known kicked top system, i.e. we deal with an angular momentum which rotates around the z-axis with a constant frequency β and is subject to periodical kicks. Every such kick revolves the top around the x-axis by an angle αJ_x. We consider Markovian dissipation chosen in such a way that the value $j(j+1)$ of the square J^2 of the angular momentum is conserved [7], therefore we deal with a finite-dimensional system. This also implies that the state of the system may be represented on a sphere of radius j. However, since it is more convenient to investigate

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the quantum–classical correspondence on a unit sphere, discussing the results we use the intensive quantities, e.g. J_z/j.

The dynamics of the density operator ρ are governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}(L_0 + L_1)\rho + \Lambda\rho \tag{1}$$

where

$$L_0\rho = \beta[J_z, \rho] \qquad L_1\rho = (\alpha/j)\sum_k [J_x^2, \rho]\delta(t-k)$$

and the dissipative part is described by

$$\Lambda\rho = \frac{\gamma_1}{2j}([J_+, \rho J_-] + \mathrm{HC}) + \frac{\gamma_2}{2j}([J_-, \rho J_+] + \mathrm{HC})\,. \tag{2}$$

In practice we are only interested in a stroboscopic map, i.e. in $\rho(t_k)$ where t_k is taken at certain determined instants between succeeding kicks, e.g. just before the kick. Then equation (1) may be solved explicitly and the quantum map reads

$$\rho_{k+1} = D\rho_k \equiv \exp(\Lambda + L_0)\exp(L_1)\rho_k\,. \tag{3}$$

Let us notice that the dissipation part of the evolution operator commutes with its free precession part, therefore the evolution operator is a product of three independent parts: free precession, dissipation and kick. In order to investigate the quantum evolution one has to compute the $(2j+1)^2 \times (2j+1)^2$ matrix D. Even for relatively small j (say $j = 20$) this requires a lot of computer memory. In the following we propose a method which avoids this numerical constraint and allows for an alternate look at the quantum chaology problems.

2. Stochastic modelling of dissipation

A description of a quantum dissipative system using the density operator ρ means that one deals with an ensemble of similar systems and all the information concerns the averaged quantities of the total ensemble. However, there appear to be some situations when one is only interested in a behaviour of an individual system interacting with an environment, e.g. in the quantum measurement problem. Searching for a theoretical description of a single act of measurement led to the formulation of some alternative quantum theories of dissipation in which the state vector represented an individual system and followed a stochastic dynamics (e.g. [8, 9], and the literature cited therein). It was argued [9, 10] that such a formulation was also applicable in the analysis of any quantum open system. Instead of looking at the evolution equation for the density operator (1) one deals with a diffusive quantum process for the state vector given by a stochastic Schrödinger equation. Such a description provides an insight into the behaviour of an individual system—it is a promising tool for investigating some fundamental problems of quantum physics (e.g. quantum jumps [10, 11]) and gives the possibility for a direct comparison with an experiment. On the other hand, and this is what we intend to stress considering a kicked top as an example, the use of a vector state instead of a density operator allows for a significant reduction of the computer storage required for calculations. Instead of using the memory size of order $(2j+1)^4$ for the evolution operator of the density matrix elements we need a size of order $(2j+1)^2$. This gives the possibility of considering much larger values of j. The price we pay for this advantage is the necessity to consider a sample large enough. However, since the size of an ensemble depends on the required precision and not on the value of j, the stochastic method thus really does establish a very effective tool for an investigation of the quantum-classical correspondence.

As is seen from (3), the dynamics of the present system may be described as the consecutive acting of a Hamiltonian kicking mechanism and of a dissipative rotation around the z-axis. Hence the stochastic formulation of the problem only concerns the latter process. Following Gisin *et al* [9, 10, 12], the diffusion equation for an unnormalized state $|\psi\rangle$ corresponding to (1) without the kicking operator L_1 reads in the Itô differential form

$$\mathrm{d}|\psi\rangle = \left[-\mathrm{i}\beta J_z + \frac{\gamma_1}{2j}(2\langle J_-\rangle J_+ - J_-J_+) + \frac{\gamma_2}{2j}(2\langle J_+\rangle J_- - J_+J_-)\right]|\psi\rangle \mathrm{d}t + \sqrt{\frac{\gamma_1}{2j}} J_+|\psi\rangle \mathrm{d}\xi_1 + \sqrt{\frac{\gamma_2}{2j}} J_-|\psi\rangle \mathrm{d}\xi_2 \,. \tag{4}$$

We deal with an unnormalized state vector since it satisfies a simpler equation than the normalized one; consequently, the quantum average value $\langle\mathcal{O}\rangle$ of an operator $\mathcal{O}$ appearing above means $\langle\psi|\mathcal{O}|\psi\rangle/\langle\psi|\psi\rangle$. The symbols ξ_1 and ξ_2 stand for the complex independent Wiener processes which satisfy the following relations:

$$\overline{\mathrm{d}\xi_k} = 0$$
$$\overline{\mathrm{Re}(\mathrm{d}\xi_k)\mathrm{Re}(\mathrm{d}\xi_l)} = \overline{\mathrm{Im}(\mathrm{d}\xi_k)\mathrm{Im}(\mathrm{d}\xi_l)} = \delta_{k,l}\,\mathrm{d}t$$
$$\overline{\mathrm{Re}(\mathrm{d}\xi_k)\mathrm{Im}(\mathrm{d}\xi_l)} = 0 \,.$$

Here the bar denotes averaging over the stochastic ensemble. For a numerical treatment we expand $|\psi\rangle$ into the complete set of eigenstates of J_z

$$|\psi\rangle = \sum_{n=-j}^{j} c_n |j, n\rangle \,.$$

Thus instead of the operator equation (4) we deal with a set of $2j+1$ complex stochastic differential equations for the coefficients $\{c_n\}$ which we solve numerically using the algorithm of Mannella [13]. The set $\{c_n\}$ was normalized after each time step of the integration procedure.

After one period the final values of $\{c_n\}$ are subject to a Hamiltonian kick which is represented by a multiplication by a known unitary matrix $\langle j, n|\exp(-\mathrm{i}(\alpha/j)J_x^2)|j, m\rangle$.

3. Results

The crucial role in the behaviour of the classical counterpart of the map (3) is played by the nonlinear parameter α [5, 6]. For small α there exists exactly one stable fixed point: the north pole for $\gamma_1 > \gamma_2$ and the south pole otherwise. At some critical value of α the north pole undergoes a pitchfork bifurcation and there appear two stable fixed points, while the south undergoes the period doubling bifurcation and one observes a stable period-two orbit. When α is further increased both solutions undergo an inverted Hopf bifurcation which results in a chaotic motion and a strange attractor appears. Although the concept of an attractor has no meaning in a quantum theory, one hopes to observe its trace in various asymptotic quantities like mean values or a density operator.

For an illustration of those different kinds of behaviour we present in figure 1 the α-dependence of the average value of J_z/j. In the classical case (full curve) we average over time, while in the quantum case (for $j = 20$, 30, and 100) it is a double average—quantum-mechanical and over the stochastic realizations. For $\alpha = 0$ the north pole is the stable point ($\gamma_1 = 0.04$ and $\gamma_2 = 0.02$) thus the classical system is localized there ($\langle J_z/j\rangle = 1$). The system undergoes a pitchfork bifurcation at $\alpha = 1$ and then there appear two new stable points which tend towards the equator as α is increased. Consequently, the value of

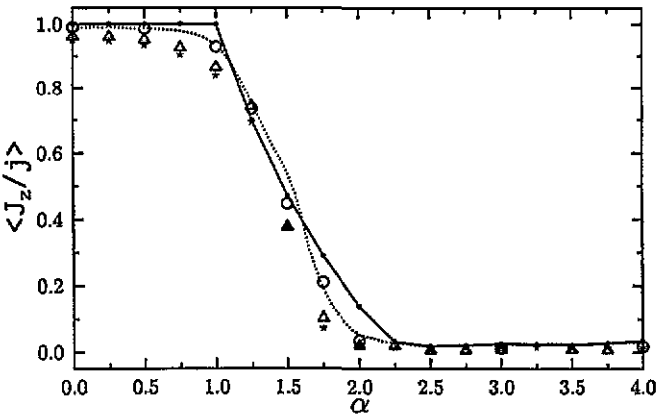

Figure 1. Classical (full curve) and quantum averages of J_z/j for $j = 20$ (stars), 30 (triangles), and 100 (circles) versus nonlinearity parameter α. The remaining parameters are: $\beta = \pi/2$, $\gamma_1 = 0.04$ and $\gamma_2 = 0.02$. The dotted curve represents the results of the classical map with addition of a small Gaussian noise. The quantum simulations were performed for 1000 realizations.

$\langle J_z/j \rangle$ becomes smaller. Finally, for $\alpha = 2.25$ a strange attractor appears. It covers the classical sphere almost uniformly, hence $\langle J_z/j \rangle$ is close to zero. (It is slightly greater than zero because we present the results just before kicking, i.e. after a dissipative part of the evolution moves the top slightly towards the north pole.)

The behaviour of quantum averages is very similar to that of the classical one. As one expects, the coincidence between the quantum and classical results improves with increasing j—for greater j the quantum state is better localized. The relatively large discrepancy in the region just before the inverted Hopf bifurcation is due to the very complicated structure of the phase space. In fact three attractors coexist: two fixed points and a strange attractor. When a basin of attraction of a fixed point is smaller than the localization region of a quantum state, the latter is related to both: a fixed point and a strange attractor. The classical case is very sensitive even for small disturbances, while due to the stochastic terms the quantum state is in a sense smoothed over. Since the amplitude of those terms is of the order of $j^{-1/2}$, the correspondence of classical and quantum averages is better for greater j. Analogously, a better coincidence is achieved if one disturbs the classical map with a stochastic term. This is illustrated in figure 1, where the dotted curve represents the results for the classical map for angular momentum components with the addition of a small uniform and isotropic Gaussian white noise over the sphere. The strength of the noise is chosen so as to approximately follow the quantum results for $j = 100$.

Another comparison of the classical and quantum counterparts of the top is displayed in figures 2–5. Namely, we compare the shape of the invariant measure $P_c(\theta, \phi)$ (θ and ϕ are the angle spherical coordinates) on the classical attractor with the shape of the Q-representation for the quantum case. To be more precise, in the case of a regular classical motion (figure 2) we also use an additional noise term in order to slightly smooth the very sharp invariant measure. In the quantum case we use the probability density function $P_q(\theta, \phi)$ for finding the quantum top in a point (θ, ϕ) on the unit sphere, which is defined as

$$P_q(\theta, \phi) = \frac{2j+1}{4\pi} \sin(\theta)\, Q(\theta, \phi)$$

where the Q-representation is the diagonal density operator element in the spin coherent

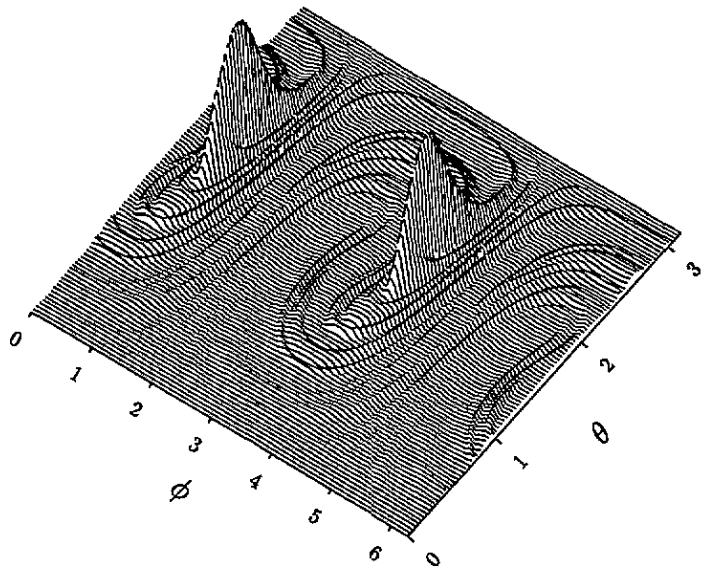

Figure 2. Classical distribution function $P_c(\theta, \phi)$ corresponding to a regular motion ($\alpha = 1.75, \beta = \pi/2, \gamma_1 = 0.04, \gamma_2 = 0.02$).

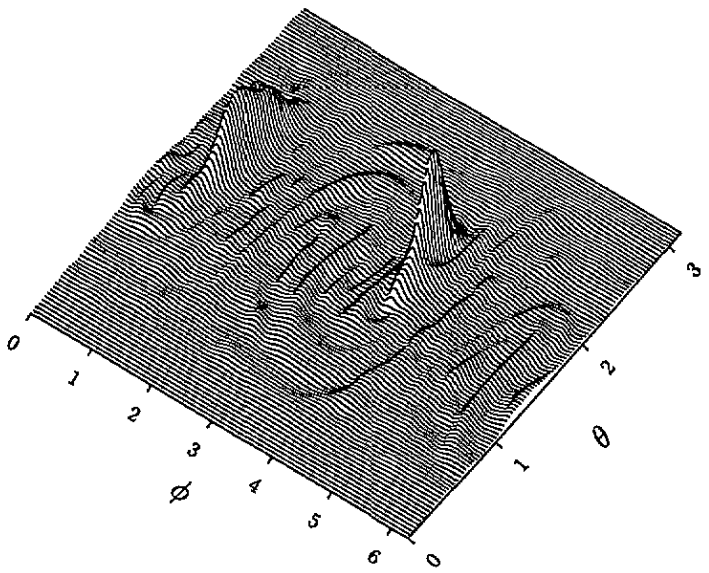

Figure 3. Quantum distribution function $P_q(\theta, \phi)$ for $j = 200$ (the other parameters are as in figure 2).

state $|\theta, \phi\rangle$ representation [14]:

$$Q(\theta, \phi) \equiv \langle \theta, \phi | \rho | \theta, \phi \rangle$$
$$= \left(\frac{|\tau|}{1 + |\tau|^2} \right)^{2j} \sum_{m,n=-j}^{j} c_m^* c_n \left(\begin{array}{c} 2j \\ j+m \end{array} \right)^{1/2} \left(\begin{array}{c} 2j \\ j+n \end{array} \right)^{1/2} \tau^m (\tau^*)^n$$

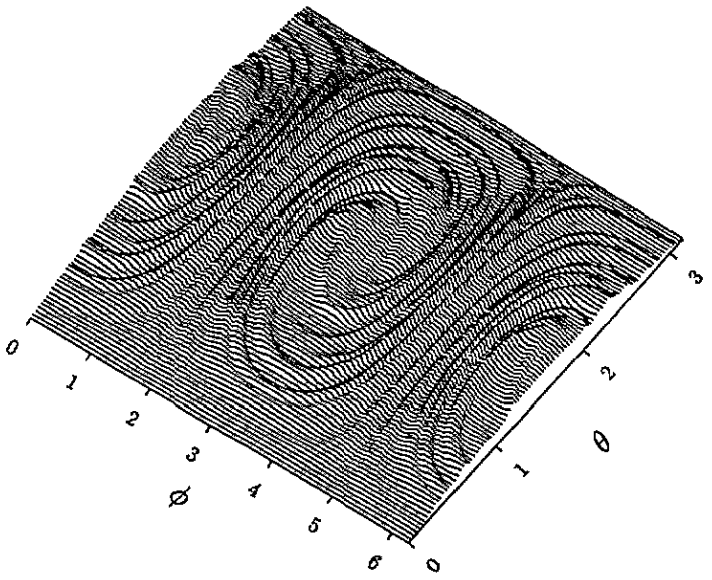

Figure 4. The same as figure 2 but for chaotic motion ($\alpha = 3$).

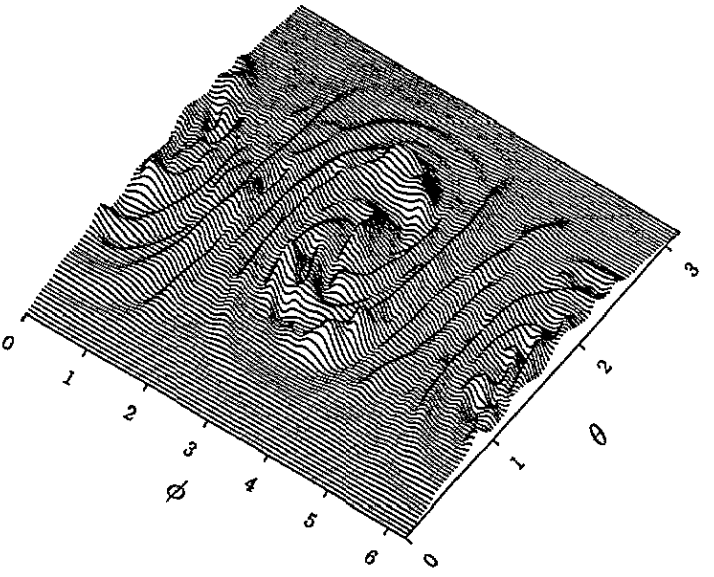

Figure 5. *The same as figure 3 but for* $\alpha = 3$.

with $\tau = \tan(\theta/2)\exp(-i\phi)$. In figures 2 and 3 we show P_c and P_q for a classically regular motion ($\alpha = 1.75$). In both cases the most probable regions on the sphere corresponding to the deterministic stable points are identified. The fold structure of the distributions reflects invariant manifolds of the stable fixed points. Figures 4 and 5 display the situation for a classically chaotic motion ($\alpha = 3.0$). Although the structure of the distributions is also very folded, there is no particularly preferred region—the chaotic state covers the sphere more or less uniformly. For both regular and chaotic motions we notice a reasonably good agreement

between the classical P_c and quantum P_q distributions, the better the agreement the greater j is—as j increases, the more subtle details of the classical attractor appear in the form of P_q.

4. Conclusions

In the present paper we investigate the relation between the classical and quantum counterparts of a damped kicked top, a system often used for discussing quantum chaology problems. We propose an alternative method for treating the dissipation mechanism by means of the stochastic Schrödinger equation (4) for the vector state $|\psi\rangle$, instead of the usually used von Neumann evolution equation (1) for the density operator ρ. Such an approach allows us to treat systems with very high values of j. The largest j used in this paper is 200, however there are no limitations for considering systems with much greater j. The essential cost we must pay for an enlargement of the size of a quantum system is the time of the numerical simulations.

The results presented demonstrate quite good agreement between the behaviour of the classical and quantum counterparts of the dissipated kicked top. This agreement becomes better when the size of the quantum top increases. This concerns both the regular and chaotic regimes of parameters. Thus the proposed method seems to be very promising while dealing with quantum counterparts of classically chaotic dissipative dynamics. We also notice that, although here the method is applied to a kicked system, it may be used directly for systems with continuous time dependence, which is practically (numerically) impossible within the framework of the density operator formalism.

Finally, we must remark that in this paper we do not treat the very important problem of a correspondence limit. The determination of the classical noise terms from the quantum mechanical foundations is especially interesting for the theory of open systems. In some cases (e.g. [3]) one uses a Gaussian (diffusive) approximation of a Wigner representation which allows one to mimic a quantum problem with a classical stochastic map. Here, we cannot follow this path since the dissipator (2) possesses a non-positive defined diffusion matrix [15] which does not allow for the Gaussian approximation. Hence the stochastic terms which we add to the classical map to qualitatively exemplify some features are not directly related to the quantum properties. We believe that Gisin's theory, being an alternative viewpoint on the behaviour of the quantum open systems, might be very suitable for searching the correspondence limit. Further investigations are in progress.

Note added in proof. After this paper had been prepared for publication Spiller and Ralph [16] presented a paper employing the theory of Gisin *et al* for treating quantum chaos. Considering a damped, driven, nonlinear oscillator they demonstrated a similarity between the Poincaré section of a classical trajectory and its quantum counterpart. While we study ensemble properties of dissipative quantum systems, intending to apply Gisin's theory to the problems previously treated by the density operator formalism, Spiller and Ralph propose a quite new, very interesting way of dealing with quantum chaos, namely an analysis of a single quantum trajectory.

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